

Pełczyński's property (V^*) of order p and its quantification[☆]

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Abstract

We introduce the concepts of Pełczyński's property (V) of order p and Pełczyński's property (V^*) of order p . It is proved that, for each $1 < p < \infty$, the James p -space J_p enjoys Pełczyński's property (V^*) of order p and the James p^* -space J_{p^*} (where p^* denotes the conjugate number of p) enjoys Pełczyński's property (V) of order p . We prove that both $L_1(\mu)$ (μ a finite positive measure) and l_1 enjoy the quantitative version of Pełczyński's property (V^*) .

Keywords: Pełczyński's property (V) of order p , Pełczyński's property (V^*) of order p , Quantifying Pełczyński's property (V^*) of order p
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1. Introduction and notations

Let X and Y be Banach spaces. Recall that an operator $T : X \rightarrow Y$ is called *unconditionally converging* if T takes weakly unconditionally Cauchy

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series in X to unconditionally converging series in Y . In his fundamental paper [21], A. Pełczyński introduced property (V). A Banach space X is said to have *Pełczyński's property (V)* if every unconditionally converging operator with domain X is unconditionally converging. Equivalently, X has Pełczyński's property (V) if a bounded subset K of X^* is relatively weakly compact whenever $\lim_{n \rightarrow \infty} \sup_{x^* \in K} |\langle x^*, x_n \rangle| = 0$ for every weakly unconditionally Cauchy series $\sum_{n=1}^{\infty} x_n$ in X . The most known classical Banach spaces that have Pełczyński's property (V) are spaces $C(\Omega)$ of continuous scalar-valued functions on compact Hausdorff space Ω [21], or more generally Banach spaces whose duals are isometric to L_1 -spaces [16]. Pełczyński's property (V^*) was introduced in [21] as a dual property of Pełczyński's property (V). A Banach space X is said to have *Pełczyński's property (V^*)* if a bounded subset K of X is relatively weakly compact whenever $\lim_{n \rightarrow \infty} \sup_{x \in K} |\langle x_n^*, x \rangle| = 0$ for every weakly unconditionally Cauchy series $\sum_{n=1}^{\infty} x_n^*$ in X^* . Among classical Banach spaces that have Pełczyński's property (V^*) , L_1 -spaces are the most notable ones.

The main goal of this paper is to generalize Pełczyński's property (V) and Pełczyński's property (V^*) to more general case. In Section 2, we introduce the concept of Pełczyński's property (V) of order p ($1 \leq p \leq \infty$) (property p -(V) in short). Property 1-(V) is precisely Pełczyński's property (V) and property ∞ -(V) is precisely the reciprocal Dunford-Pettis property (see [18] for this definition). It is clear that for each $1 < p < \infty$, a Banach space X has property p -(V) whenever X has Pełczyński's property (V). It is natural to ask whether there exists a space enjoying property p -(V) fails Pełczyński's property (V) for each $1 < p < \infty$. In this section, we show that for each $1 < p < \infty$, the James p^* -space J_{p^*} (where p^* denotes the conjugate number of p) has property p -(V) (see Theorem 2.6 below). But, J_{p^*} clearly fails Pełczyński's property (V) because J_{p^*} contains no copy of c_0 and is non-reflexive. It is proved in [9] that Pełczyński's property (V) is not a three-space property, that is, there exist a space X failing Pełczyński's property (V) and a closed subspace X_0 of X such that both X_0 and the quotient X/X_0 have Pełczyński's property (V). We extend this result to property p -(V) and show that property p -(V) are not three-space properties for each $1 \leq p < \infty$. The concept of Pełczyński's property (V^*) of order p (property p -(V^*) in short) is introduced in this section. Property 1-(V^*) is precisely Pełczyński's property (V^*) . Similarly, for each $1 < p < \infty$, a Banach space X has property p -(V^*) whenever X has Pełczyński's property (V^*) . We show that the converse is false. For each $1 < p < \infty$, the James p -space J_p fails

Pelczyński's property (V^*) because J_p is not weakly sequentially complete. But J_p enjoys property $p\text{-}(V^*)$ for each $1 < p < \infty$ as shown in the following Theorem 2.11. In [21], A. Pelczyński proved that if a Banach space X has both property (V) and property (V^*) , then X must be reflexive. However, Theorem 2.6 and Theorem 2.11 in this section tell us that the classical non-reflexive James space J has both property $2\text{-}(V)$ and property $2\text{-}(V^*)$. A. Pelczyński showed in [21] that if a Banach space X has property (V^*) , then X must be weakly sequentially complete. Correspondingly, we introduce the notion of weak sequential completeness of order p and show that if a Banach space X has property $p\text{-}(V^*)$, then X must be weakly sequentially complete of order p for each $1 < p < 2$.

In [5], F. Bombal studied Pelczyński's property (V^*) in vector-valued sequence spaces and proved that given a sequence $(X_n)_n$ of Banach spaces, the space $(\sum_{n=1}^{\infty} \oplus X_n)_p$ ($1 \leq p < \infty$) has Pelczyński's property (V^*) if and only if each X_n does. Our Theorem 3.9 and Theorem 3.10 in Section 3 cover this result. Moreover, we characterize $p\text{-}(V)$ sets and prove that the space $(\sum_{n=1}^{\infty} \oplus X_n)_p$ ($1 < p < \infty$) has property $q\text{-}(V)$ ($1 \leq q < \infty$) if and only if each X_n does. In particular, we show that the space $(\sum_{n=1}^{\infty} \oplus X_n)_p$ ($1 < p < \infty$ or $p = 0$) has Pelczyński's property (V) if and only if each X_n does.

Section 3 is concerned with quantifications of property $p\text{-}(V^*)$ and property $p\text{-}(V)$. H. Krulišová [19] introduced several possibilities of quantifying Pelczyński's property (V) and proved a quantitative version of Pelczyński's result about $C(K)$ spaces. More precisely, he proved that the space $C_0(\Omega)$ enjoys the quantitative property $(V_q)_\omega^*$ with constant π (2 in the real case) for every locally compact Hausdorff space Ω . In this section, we introduce the concepts of quantitative Pelczyński's property (V^*) of order p and quantitative Pelczyński's property (V) of order p . First we prove quantitative versions of some results about property $p\text{-}(V^*)$ and property $p\text{-}(V)$. It is proved in [17] that the quantities $\omega(\cdot)$ and $wk(\cdot)$ are equal in $L_1(\mu)$ for a general positive measure μ . In this section, we introduce a quantity $\iota_p(\cdot)$ ($1 \leq p < \infty$) and prove that the quantities $wk(\cdot)$ and ι_1 are equal in $L_1(\mu)$ (μ a finite positive measure) and l_1 . In particular, both $L_1(\mu)$ (μ a finite positive measure) and l_1 have quantitative Pelczyński's property (V^*) with constant 1. Finally, we show that c_0 enjoys the quantitative property $(V_q)_\omega^*$ with constant 1.

Our notation and terminology are standard as may be found in [4] and [20]. Throughout the paper, all Banach spaces can be considered either real or complex unless stated otherwise. By an operator, we always mean a bounded linear operator. p^* will always denote the conjugate number of p for

$1 \leq p < \infty$. Let X be a Banach space, $1 \leq p < \infty$ and we denote $l_p^w(X)$ by the space of all weakly p -summable sequences in X , endowed with the norm

$$\|(x_n)_n\|_p^w = \sup\left\{\left(\sum_{n=1}^{\infty} |\langle x^*, x_n \rangle|^p\right)^{\frac{1}{p}} : x^* \in B_{X^*}\right\}, \quad (x_n)_n \in l_p^w(X).$$

A sequence $(x_n)_n \in l_p^w(X)$ is *unconditionally p -summable* if

$$\sup\left\{\left(\sum_{n=m}^{\infty} |\langle x^*, x_n \rangle|^p\right)^{\frac{1}{p}} : x^* \in B_{X^*}\right\} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

In [7], we extend unconditionally converging operators and completely continuous operators to the general case $1 \leq p \leq \infty$. Let $1 \leq p \leq \infty$. We say that an operator $T : X \rightarrow Y$ is *unconditionally p -converging* if T takes weakly p -summable sequences (weakly null sequences for $p = \infty$) to unconditionally p -summable sequences (norm null sequences for $p = \infty$).

2. Pełczyński's property (V) of order p and Pełczyński's property (V^*) of order p

Definition 2.1. Let $1 \leq p \leq \infty$. We say that a Banach space X has *Pełczyński's property (V) of order p* (property p -(V) in short) if for every Banach space Y , every unconditionally p -converging operator $T : X \rightarrow Y$ is weakly compact.

Obviously, for every $1 \leq p < q \leq \infty$, a Banach space X has property q -(V) whenever X has property p -(V).

Definition 2.2. [7] Let X be a Banach space and $1 \leq p \leq \infty$. We say that a bounded subset K of X^* is a *p -(V) set* if

$$\lim_{n \rightarrow \infty} \sup_{x^* \in K} |\langle x^*, x_n \rangle| = 0,$$

for every $(x_n)_n \in l_p^w(X)$ ($(x_n)_n \in c_0^w(X)$ for $p = \infty$).

1-(V) sets are called (V)-sets in [8] and ∞ -(V) sets are called (L)-sets (see [11] for example). Before giving a useful characterization of a p -(V) set, we recall the notion of weakly p -convergent sequences introduced in [10]. Let $1 \leq p \leq \infty$. A sequence $(x_n)_n$ in a Banach space X is said to be

weakly p -convergent to $x \in X$ if the sequence $(x_n - x)_n$ is weakly p -summable in X . Weakly ∞ -convergent sequences are simply the weakly convergent sequences. The concept of weakly p -Cauchy sequences is introduced in [7]. We say that a sequence $(x_n)_n$ in a Banach space X is *weakly p -Cauchy* if for each pair of strictly increasing sequences $(k_n)_n$ and $(j_n)_n$ of positive integers, the sequence $(x_{k_n} - x_{j_n})_n$ is weakly p -summable in X . Obviously, every weakly p -convergent sequence is weakly p -Cauchy, and the weakly ∞ -Cauchy sequences are precisely the weakly Cauchy sequences. J.M.F.Castillo and F.Sánchez said that a Banach space $X \in W_p(1 \leq p \leq \infty)$ if any bounded sequence in X admits a weakly p -convergent subsequence (see [10]). The following characterization of a p -(V) set appears in [7].

Theorem 2.1. [7] *Let $1 < p < \infty$ and X be a Banach space. The following statements are equivalent about a bounded subset K of X^* :*

- (1) *K is a p -(V) set;*
- (2) *For all spaces $Y \in W_p$ and for every operator T from Y into X , the subset $T^*(K)$ is relatively norm compact;*
- (3) *For every operator T from l_{p^*} into X , the subset $T^*(K)$ is relatively norm compact.*

In case of $p = 1$, R. Cilia and G. Emmanuele proved that a bounded subset K of X^* is a 1-(V) set if and only if for every operator T from c_0 into X , the subset $T^*(K)$ is relatively norm compact (see [8]). However, the equivalence between (1) and (3) of Theorem 2.1 is false for $p = \infty$ ($p^* = 1$). For instance, by Schur property, every bounded subset of l_∞ is a ∞ -(V) set, but B_{l_∞} is not relatively norm compact.

Let us fix some notations. If A and B are nonempty subsets of a Banach space X , we set

$$d(A, B) = \inf\{\|a - b\| : a \in A, b \in B\},$$

$$\widehat{d}(A, B) = \sup\{d(a, B) : a \in A\}.$$

Thus, $d(A, B)$ is the ordinary distance between A and B , and $\widehat{d}(A, B)$ is the non-symmetrized Hausdorff distance from A to B .

Let X be a Banach space and A be a bounded subset of X^* . For $1 \leq p \leq \infty$, we set

$$\xi_p(A) = \inf\{\widehat{d}(A, K) : K \subset X^* \text{ is a } p\text{-(}V\text{) set}\}.$$

It is clear that $\xi_p(A) = 0$ if and only if A is a p -(V) set.

Let A be a bounded subset of a Banach space X . The de Blasi measure of weak non-compactness of A is defined by

$$\omega(A) = \inf\{\widehat{d}(A, K) : \emptyset \neq K \subset X \text{ is weakly compact}\}.$$

Then $\omega(A) = 0$ if and only if A is relatively weakly compact. For an operator $T : X \rightarrow Y$, we denote $\xi_p(TB_X), \omega(TB_X)$ by $\xi_p(T), \omega(T)$ respectively.

To characterize property p -(V), we need the following result in [7].

Theorem 2.2. [7] *Let $1 \leq p \leq \infty$. The following statements about an operator $T : X \rightarrow Y$ are equivalent:*

- (1) *T is unconditionally p -converging;*
- (2) *T sends weakly p -summable sequences onto norm null sequences;*
- (3) *T sends weakly p -Cauchy sequences onto norm convergent sequences.*

The following result shows that Pełczyński's property (V) of order p is automatically quantitative in some sense.

Theorem 2.3. *Let $1 \leq p \leq \infty$ and X be a Banach space. The following are equivalent:*

- (1) *X has property p -(V);*
- (2) *Every p -(V) subset of X^* is relatively weakly compact;*
- (3) *$\omega(T^*) \leq \xi_p(T^*)$ for every operator T from X into any Banach space Y ;*
- (4) *$\omega(A) \leq \xi_p(A)$ for every bounded subset A of X^* .*

Proof. (1) \Rightarrow (2). Suppose that K is a p -(V) subset of X^* . Take any sequence $(x_n^*)_n$ in K . Define an operator $T : X \rightarrow l_\infty$ by

$$Tx = (\langle x_n^*, x \rangle)_n, \quad x \in X.$$

Then, for every $(x_n)_n \in l_p^w(X)$, we have

$$\|Tx_n\| = \sup_k |\langle x_k^*, x_n \rangle| \leq \sup_{x^* \in K} |\langle x^*, x_n \rangle| \rightarrow 0 \quad (n \rightarrow \infty).$$

It follows from Theorem 2.2 that T is unconditionally p -converging. By (1), the operator T is weakly compact and hence T^* is also weakly compact. This implies that the set $T^*B_{l_\infty}$ is relatively weakly compact. It is easy to see that

$T^*e_n = x_n^*$ for each $n \in \mathbb{N}$, where $(e_n)_n$ is the unit vector basis of l_1 . So the sequence $(x_n^*)_n$ is relatively weakly compact.

(2) \Rightarrow (3) is obvious. (3) \Rightarrow (1) is immediate from Theorem 2.2. The equivalence of (2) \Leftrightarrow (4) is straightforward. \square

Corollary 2.4. *Let $1 \leq p \leq \infty$. If a Banach space X has property p -(V), then every quotient of X has property p -(V).*

Recall that the James p -space J_p ($1 < p < \infty$) is the (real) Banach space of all sequences $(a_n)_n$ of real numbers such that $\lim_{n \rightarrow \infty} a_n = 0$ and

$$\begin{aligned} \|(a_n)_n\|_{cpv} &= \frac{1}{2^{\frac{1}{p}}} \sup \left\{ \left(\sum_{j=1}^m |a_{i_{j-1}} - a_{i_j}|^p + |a_{i_m} - a_{i_0}|^p \right)^{\frac{1}{p}} : 1 \leq i_0 < i_1 < \dots < i_m, m \in \mathbb{N} \right\} \\ &< \infty. \end{aligned}$$

Another useful equivalent norm on J_p is given by the formula

$$\|(a_n)_n\|_{pv} = \sup \left\{ \left(\sum_{j=1}^m |a_{i_{j-1}} - a_{i_j}|^p \right)^{\frac{1}{p}} : 1 \leq i_0 < i_1 < \dots < i_m, m \in \mathbb{N} \right\}.$$

In fact,

$$\frac{1}{2^{\frac{1}{p}}} \|\cdot\|_{pv} \leq \|\cdot\|_{cpv} \leq \|\cdot\|_{pv}. \quad (2.1)$$

The sequence $(e_n)_n$ of standard unit vectors forms a monotone shrinking basis for J_p in both norms $\|\cdot\|_{pv}$ and $\|\cdot\|_{cpv}$. It is known that J_p is non-reflexive and is codimension of 1 in J_p^{**} , but every infinite-dimensional closed subspace of J_p contains a subspace isomorphic to l_p .

The following lemma may appear somewhere. Its proof is identical to [4, Proposition 3.4.3].

Lemma 2.5. *Let $(x_k)_k$ be a normalized block basic sequence with respect to $(e_n)_n$ in $(J_p, \|\cdot\|_{pv})$. Then, for any sequence $(\lambda_k)_{k=1}^n$ of real numbers and any $n \in \mathbb{N}$ the following estimate holds:*

$$\left\| \sum_{k=1}^n \lambda_k x_k \right\|_{pv} \leq (1 + 2^p)^{\frac{1}{p}} \left(\sum_{k=1}^n |\lambda_k|^p \right)^{\frac{1}{p}}.$$

Theorem 2.6. *The James p -space J_p has property p^* -(V).*

Proof. Let K be a p^* -(V) subset of $B_{J_p^*}$. Take any sequence $(x_n^*)_n$ from K . Since J_p is separable, we may assume that $(x_n^*)_n$ is *weak**-convergent to some $x_0^* \in J_p^*$.

Claim. $(x_n^*)_n$ converges to x_0^* weakly.

Note that $J_p^{**} = J_p \oplus \text{span}\{x_0^{**}\}$, where $x_0^{**} \in J_p^{**}$ is defined by $\langle x_0^{**}, e_n^* \rangle = 1$ for all $n \in \mathbb{N}$, where $(e_n^*)_n$ is the functionals biorthogonal to the unit vector basis $(e_n)_n$ of J_p . Thus it suffices to prove that

$$\langle x_0^{**}, x_n^* \rangle \rightarrow \langle x_0^{**}, x_0^* \rangle \quad (n \rightarrow \infty).$$

Suppose it is false. By passing to subsequences, we may assume that $|\langle x_0^{**}, x_n^* - x_0^* \rangle| > \epsilon_0$ for some $\epsilon_0 > 0$ and for all $n \in \mathbb{N}$. Since $(e_n)_n$ is shrinking, $(e_n^*)_n$ forms a basis for J_p^* . Thus

$$\left| \sum_{k=1}^{\infty} \langle x_n^* - x_0^*, e_k \rangle \right| = \left| \langle x_0^{**}, \sum_{k=1}^{\infty} \langle x_n^* - x_0^*, e_k \rangle e_k^* \rangle \right| > \epsilon_0, \quad n = 1, 2, \dots \quad (2.2)$$

Note that

$$\lim_{n \rightarrow \infty} \langle x_n^* - x_0^*, e_k \rangle = 0, \quad k = 1, 2, \dots \quad (2.3)$$

By inductions on n in (2.2) and on k in (2.3), we obtain $1 = n_1 < n_2 < n_3 < \dots$ and $0 = k_0 < k_1 < k_2 < k_3 < \dots$ such that

$$\left| \sum_{k=k_{j-1}+1}^{k_j} \langle x_{n_j}^* - x_0^*, e_k \rangle \right| > \frac{\epsilon_0}{2}, \quad j = 1, 2, \dots \quad (2.4)$$

We set $x_j = \sum_{k=k_{j-1}+1}^{k_j} e_k$ ($j = 1, 2, \dots$). Then $(x_j)_j$ is a normalized block basic sequence with respect to $(e_n)_n$ in $(J_p, \|\cdot\|_{cpv})$. It follows from (2.1) and Lemma 2.5 that for any sequence of real scalars $(\lambda_j)_{j=1}^n$ the following estimate holds

$$\left\| \sum_{j=1}^n \lambda_j x_j \right\|_{cpv} \leq (2 + 2^{p+1})^{\frac{1}{p}} \left(\sum_{j=1}^n |\lambda_j|^p \right)^{\frac{1}{p}}$$

This implies that $(x_j)_j$ is weakly p^* -summable. Since K is a p^* -(V) set, we get

$$|\langle x_{n_j}^*, x_j \rangle| \leq \sup_{x^* \in K} |\langle x^*, x_j \rangle| \rightarrow 0 \quad (j \rightarrow \infty).$$

Obviously, $(\langle x_0^*, x_j \rangle)_j$ converges to 0. Therefore, we have

$$|\langle x_{n_j}^* - x_0^*, x_j \rangle| \rightarrow 0 \quad (j \rightarrow \infty),$$

which contradicts with (2.4). This contradiction shows that $(x_n^*)_n$ converges to x_0^* weakly. Thus K is relatively weakly compact. By Theorem 2.3, J_p has property p^* -(V). □

As in [9], we consider the space X_p constructed in [13]. We do not describe the space X_p here and refer the reader to [13] for details. In [13], a quotient map $T_p : X_p \rightarrow c_0$ is defined and it is proved that T_p is unconditionally converging. We extend this result as follows:

Lemma 2.7. *For $1 < p < \infty$, the quotient map T_p is unconditionally q -converging for any $1 \leq q < p^*$.*

Proof. Suppose that T_p is not unconditionally q -converging for some $1 \leq q < p^*$. Then there exists an operator S from l_{q^*} (c_0 for $q = 1$) into X_p such that $T_p S$ is non-compact. Thus, we can find a weakly null sequence $(z_n)_n$ in l_{q^*} and $\epsilon_0 > 0$ such that $\|T_p S z_n\| \geq \epsilon_0$ for each $n \in \mathbb{N}$. By passing to subsequences, we may assume that $(z_n)_n$ is equivalent to the unit vector basis of l_{q^*} , that is, there exist $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$ and all scalars $\alpha_1, \alpha_2, \dots, \alpha_n$, one has

$$C_1 \left(\sum_{k=1}^n |\alpha_k|^{q^*} \right)^{\frac{1}{q^*}} \leq \left\| \sum_{k=1}^n \alpha_k z_k \right\| \leq C_2 \left(\sum_{k=1}^n |\alpha_k|^{q^*} \right)^{\frac{1}{q^*}}. \quad (2.5)$$

Let $x_n = S z_n$. By [14, Proposition 2], the sequence $(x_n)_n$ admits a subsequence, which is still denoted by $(x_n)_n$, such that $(x_{2n-1} - x_{2n})_n$ is equivalent to the unit vector basis of l_p . Then, there exist $D_1, D_2 > 0$ such that for all $n \in \mathbb{N}$ and all scalars $\alpha_1, \alpha_2, \dots, \alpha_n$, one has

$$D_1 \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \leq \left\| \sum_{k=1}^n \alpha_k (x_{2k-1} - x_{2k}) \right\| \leq D_2 \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}}. \quad (2.6)$$

By (2.5) and (2.6), we get, for each n and scalars $\alpha_1, \alpha_2, \dots, \alpha_n$,

$$\begin{aligned} D_1 \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} &\leq \left\| \sum_{k=1}^n \alpha_k (x_{2k-1} - x_{2k}) \right\| \\ &\leq \|S\| \cdot \left\| \sum_{k=1}^n \alpha_k (z_{2k-1} - z_{2k}) \right\| \\ &\leq \|S\| \cdot C_2 \cdot 2^{\frac{1}{q^*}} \left(\sum_{k=1}^n |\alpha_k|^{q^*} \right)^{\frac{1}{q^*}}, \end{aligned}$$

which is impossible because $1 \leq q < p^*$. This completes the proof. \square

Theorem 2.8. *Property q -(V) is not a three-space property for each $1 \leq q < \infty$.*

Proof. For $1 \leq q < \infty$, choose $1 < p < \infty$ with $q < p^*$. It is shown in [9] that both $X_p/\text{Ker}(T_p)$ and $\text{Ker}(T_p)$ have property 1-(V) and hence have property q -(V). But X_p fails property q -(V) since, by Lemma 2.7, T_p is unconditionally q -converging, but obviously not weakly compact. \square

Definition 2.3. Let X be a Banach space and $1 \leq p \leq \infty$. We say that a bounded subset K of X is a p -(V^*) set if

$$\lim_{n \rightarrow \infty} \sup_{x \in K} | \langle x_n^*, x \rangle | = 0,$$

for every $(x_n^*)_n \in l_p^w(X^*)$ ($(x_n^*)_n \in c_0^w(X^*)$ for $p = \infty$).

It is noted that 1-(V^*) sets are (V^*)-sets (see [21]) and ∞ -(V^*) sets are Dunford-Pettis sets.

Theorem 2.9. *Let K be a bounded subset of a Banach space X and $1 < p < \infty$. The following statements are equivalent:*

- (1) K is a p -(V^*) set;
- (2) For all spaces Y with $Y^* \in W_p$, every operator $T : X \rightarrow Y$ maps K onto a relatively norm compact subset of Y ;
- (3) Every operator $T : X \rightarrow l_p$ maps K onto a relatively norm compact subset of l_p .

Proof. (1) \Rightarrow (2). Let Y and T be as stated in (2). Assume the contrary that $T(K)$ is not relatively norm compact. Then there exists a sequence $(x_n)_n$ in K such that $(Tx_n)_n$ admits no norm convergent subsequences. Since Y is reflexive, by passing to a subsequence if necessary we may assume that $(Tx_n)_n$ converges weakly to some $y \in Y$ and $\|Tx_n - y\| > \epsilon_0$ for some $\epsilon_0 > 0$ and for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, choose y_n^* with $\|y_n^*\| \leq 1$ such that $|\langle y_n^*, Tx_n - y \rangle| > \epsilon_0$. Since $Y^* \in W_p$, by passing to a subsequence again one can assume that the sequence $(y_n^*)_n$ is weakly p -convergent to some $y^* \in Y^*$. By (1), we get

$$\lim_{n \rightarrow \infty} \sup_{x \in K} |\langle T^* y_n^* - T^* y^*, x \rangle| = 0.$$

For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \epsilon_0 &< |\langle y_n^*, Tx_n - y \rangle| \\ &\leq |\langle y_n^*, Tx_n \rangle - \langle y^*, Tx_n \rangle| + |\langle y^*, Tx_n \rangle - \langle y^*, y \rangle| \\ &\quad + |\langle y^*, y \rangle - \langle y_n^*, y \rangle| \\ &\leq \sup_{x \in K} |\langle T^* y_n^* - T^* y^*, x \rangle| + |\langle y^*, Tx_n - y \rangle| \\ &\quad + |\langle y^* - y_n^*, y \rangle| \rightarrow 0 \quad (n \rightarrow \infty), \end{aligned}$$

which is a contradiction.

(2) \Rightarrow (3) is immediate because $l_{p^*} \in W_p$;

(3) \Rightarrow (1). Let $(x_n^*)_n \in l_p^w(X^*)$. Then there exists an operator T from X into l_p such that $Tx = (\langle x_n^*, x \rangle)_n$ for all $x \in X$. It follows from (3) that $T(K)$ is relatively norm compact. By the well-known characterization of relatively norm compact subsets of l_p , one can derive that $\lim_{n \rightarrow \infty} \sup_{x \in K} |\langle x_n^*, x \rangle| = 0$. This finishes the proof. \square

It should be mentioned that G. Emmanuele proved the equivalence between (1) and (3) of Theorem 2.9 for $p = 1$ (see [12]). Obviously, this is false for $p = \infty$, for example, take $X = c_0$. But, K. T. Andrews proved that a bounded subset K of a Banach space X is a ∞ -(V^*) set if and only if every weakly compact operator $T : X \rightarrow c_0$ maps K onto a relatively norm compact subset (see [2]).

Let X be a Banach space and A be a bounded subset of X . For $1 \leq p \leq \infty$, we set

$$\theta_p(A) = \inf\{\widehat{d}(A, K) : K \subset X \text{ is a } p\text{-}(V^*) \text{ set}\}.$$

Obviously, $\theta_p(A) = 0$ if and only if A is a $p\text{-}(V^*)$ set.

Definition 2.4. Let $1 \leq p \leq \infty$. We say that a Banach space X has *Pełczyński's property (V^*) of order p* ($p\text{-}(V^*)$ in short) if every $p\text{-}(V^*)$ subset of X is relatively weakly compact.

It is clear that for every $1 \leq p < q \leq \infty$, a Banach space X has property $q\text{-}(V^*)$ whenever X has property $p\text{-}(V^*)$.

The proof of the following lemma is similar to [1, Proposition 5].

Lemma 2.10. Let $(x_n^*)_n = (\sum_{i=k_{n-1}+1}^{k_n} a_i e_i^*)_n$ be a semi-normalized block basic sequence with respect to $(e_n^*)_n$ in J_p^* and suppose that $\sum_{i=k_{n-1}+1}^{k_n} a_i = 0$ for each $n \in \mathbb{N}$. Then $(x_n^*)_n$ is equivalent to the unit vector basis of l_{p^*} .

Theorem 2.11. The James p -space J_p has property $p\text{-}(V^*)$.

Proof. Let K be a $p\text{-}(V^*)$ subset of B_{J_p} . Take any sequence $(x_n)_n$ from K . Since J_p^* is separable, we may assume that $(x_n)_n$ is *weak**-convergent to some $x^{**} \in B_{J_p^{**}}$. It aims to prove that $x^{**} \in J_p$, that is, $\lim_{k \rightarrow \infty} \langle x^{**}, e_k^* \rangle = \xi = 0$.

Suppose that $\xi \neq 0$. Let $\delta = \frac{|\xi|}{2} > 0$. Then there exists $p_1 \in \mathbb{N}$ such that $|\langle x^{**}, e_k^* \rangle| > \delta$ for all $k \geq p_1$. Since $(x_n)_n$ is *weak**-convergent to x^{**} , we choose n_1 such that $|\langle e_{p_1}^*, x_{n_1} \rangle| > \delta$. Choose $q_1 > p_1$ such that $|\langle e_k^*, x_{n_1} \rangle| < \frac{\delta}{2}$ for each $k \geq q_1$. In particular, $|\langle e_{q_1}^*, x_{n_1} \rangle| < \frac{\delta}{2}$. Choose any $p_2 > q_1$. Then there exists $n_2 > n_1$ such that $|\langle e_{p_2}^*, x_{n_2} \rangle| > \delta$. Choose $q_2 > p_2$ such that $|\langle e_k^*, x_{n_2} \rangle| < \frac{\delta}{2}$ for each $k \geq q_2$. In particular, $|\langle e_{q_2}^*, x_{n_2} \rangle| < \frac{\delta}{2}$. We continue in a similar manner and obtain

$$p_1 < q_1 < p_2 < q_2 < \cdots \text{ and } n_1 < n_2 < \cdots$$

such that

$$|\langle e_{p_j}^*, x_{n_j} \rangle| > \delta \text{ and } |\langle e_{q_j}^*, x_{n_j} \rangle| < \frac{\delta}{2}, \quad j = 1, 2, \dots$$

Set $z_j^* = e_{p_j}^* - e_{q_j}^*$ ($j = 1, 2, \dots$). Then, for each $j \in \mathbb{N}$, we have

$$\begin{aligned} |\langle z_j^*, x_{n_j} \rangle| &= |\langle e_{p_j}^*, x_{n_j} \rangle - \langle e_{q_j}^*, x_{n_j} \rangle| \\ &\geq |\langle e_{p_j}^*, x_{n_j} \rangle| - |\langle e_{q_j}^*, x_{n_j} \rangle| \\ &> \delta - \frac{\delta}{2} = \frac{\delta}{2}. \end{aligned}$$

Thus $(z_j^*)_j$ is a semi-normalized block basic sequence of $(e_n^*)_n$. It follows from Lemma 2.10 that $(z_j^*)_j$ is equivalent to the unit vector basis of l_p^* . In particular, $(z_j^*)_j$ is weakly p -summable. Since K is a p -(V^*) set, we get

$$\frac{\delta}{2} < |\langle z_j^*, x_{n_j} \rangle| \leq \sup_{x \in K} |\langle z_j^*, x \rangle| \rightarrow 0 \quad (j \rightarrow \infty),$$

which is a contradiction. □

The proof of the following theorem is similar to Theorem 2.3.

Theorem 2.12. *Let $1 \leq p \leq \infty$ and X be a Banach space. The following are equivalent:*

- (1) X has property p -(V^*);
- (2) For all spaces Y , an operator $T : Y \rightarrow X$ is weakly compact whenever T^* is unconditionally p -converging;
- (3) $\omega(T) \leq \theta_p(T)$ for every operator T from any Banach space Y into X ;
- (4) $\omega(A) \leq \theta_p(A)$ for every bounded subset A of X .

Corollary 2.13. *Let $1 \leq p \leq \infty$. If a Banach space X has property p -(V^*), then every closed subspace of X has property p -(V^*).*

Corollary 2.14. *Let $1 \leq p \leq \infty$ and X be a Banach space. Then*

- (1) *If X has property p -(V), then X^* has property p -(V^*);*
- (2) *If X^* has property p -(V), then X has property p -(V^*).*

We remark that the converse of Corollary 2.14 is not true for all $1 \leq p \leq \infty$. J. Bourgain and F. Delbaen (see [6]) constructed a Banach space X_{BD} such that X_{BD} has the Schur property, X_{BD}^* is isomorphic to an L_1 -space. Thus, the space X_{BD} fails property p -(V) for all $1 \leq p \leq \infty$. Since X_{BD}^* is isomorphic to an L_1 -space, X_{BD}^* has property 1-(V^*) and hence property p -(V^*) for all $1 \leq p \leq \infty$.

Definition 2.5. Let $1 \leq p \leq \infty$. We say that a Banach space X is *weakly sequentially complete of order p* if every weakly p -Cauchy sequence in X is weakly p -convergent.

The weakly sequential completeness of order ∞ is precisely the classical weakly sequential completeness. It is easy to verify that for $1 \leq p < q \leq \infty$, a Banach space X is weakly sequentially complete of order p whenever X is weakly sequentially complete of order q .

Theorem 2.15. *Let $1 < p < 2$. If a Banach space X has property p -(V^*), then X is weakly sequentially complete of order p .*

Proof. It follows from $1 < p < 2$ that the identity $I_p : l_p \rightarrow l_p$ is unconditionally p -converging. By Theorem 2.2, we see that every weakly p -Cauchy sequence in l_p is convergent in norm. Let $(x_n)_n$ be a weakly p -Cauchy sequence in X . Then, for every operator $T : X \rightarrow l_p$, the sequence $(Tx_n)_n$ is weakly p -Cauchy and hence is convergent in norm. By Theorem 2.9, we get that $(x_n)_n$ is a p -(V^*) set. Since X has property p -(V^*), the sequence $(x_n)_n$ is relatively weakly compact. Thus, $(x_n)_n$ is weakly p -convergent. \square

Corollary 2.16. *Let $1 < p < 2$. If a Banach space X has property p -(V), then X^* is weakly sequentially complete of order p .*

3. Pełczyński's property (V) of order p and Pełczyński's property (V^*) of order p in vector-valued sequence spaces

Let $(X_n)_n$ be a sequence of Banach spaces and $1 \leq p < \infty$. We denote by $(\sum_{n=1}^{\infty} \oplus X_n)_p$ the space of all vector-valued sequences $x = (x_n)_n$ with $x_n \in X_n (n \in \mathbb{N})$, for which

$$\|x\| = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} < \infty.$$

Similarly, $(\sum_{n=1}^{\infty} \oplus X_n)_0$ denotes the space of all vector-valued sequences $x = (x_n)_n$ with $x_n \in X_n (n \in \mathbb{N})$, for which $\lim_{n \rightarrow \infty} \|x_n\| = 0$, endowed with the supreme norm. The direct sum in the sense of l_{∞} of $(X_n)_n$, denoted by $(\sum_{n=1}^{\infty} \oplus X_n)_{\infty}$, is defined in an analogous way. For every $n \in \mathbb{N}$, I_n will denote the canonical injection from X_n into $(\sum_{n=1}^{\infty} \oplus X_n)_p$ and π_n will denote the canonical projection from $(\sum_{n=1}^{\infty} \oplus X_n)_p$ into X_n . We denote the canonical injection J_n from X_n^* into $(\sum_{n=1}^{\infty} \oplus X_n^*)_{p^*}$ and the canonical projection from $(\sum_{n=1}^{\infty} \oplus X_n^*)_{p^*}$ onto X_n^* by P_n . Clearly, $I_n^* = P_n$ and $\pi_n^* = J_n$.

Theorem 3.1. *Let $(X_n)_n$ be a sequence of Banach spaces and let $X = (\sum_{n=1}^{\infty} \oplus X_n)_p (1 < p < \infty)$ or $X = (\sum_{n=1}^{\infty} \oplus X_n)_0$. The following are equivalent for a bounded subset A of X^* :*

- (1) A is a p^* -(V) set;
(2) $P_n(A)$ is a p^* -(V) set for each $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \sup \left\{ \sum_{k=n}^{\infty} \|P_k x^*\|^{p^*} : x^* \in A \right\} = 0.$$

Proof. (1) \Rightarrow (2). It is obvious that $P_n(A)$ is a p^* -(V) set for each $n \in \mathbb{N}$. Let us assume that

$$\lim_{n \rightarrow \infty} \sup \left\{ \sum_{k=n}^{\infty} \|P_k x^*\|^{p^*} : x^* \in A \right\} \neq 0.$$

By induction, we can find $\epsilon_0 > 0$, two sequences of positive integers $(p_n)_n, (q_n)_n$ with $p_n < q_n < p_{n+1}$ ($n \in \mathbb{N}$) and a sequence $(x_n^*)_n$ in A such that $\sum_{k=p_n}^{q_n} \|P_k x_n^*\|^{p^*} > \epsilon_0$ for each $n \in \mathbb{N}$. By Hahn-Banach Theorem, for each $n \in \mathbb{N}$, there exists a sequence $(x_k^{(n)})_{k=p_n}^{q_n} \in (\sum_{k=p_n}^{q_n} \oplus X_k)_p$ such that $\sum_{k=p_n}^{q_n} \|x_k^{(n)}\|^p = 1$ and

$$\sum_{k=p_n}^{q_n} \langle P_k x_n^*, x_k^{(n)} \rangle = \left(\sum_{k=p_n}^{q_n} \|P_k x_n^*\|^{p^*} \right)^{\frac{1}{p^*}} > \epsilon_0^{\frac{1}{p^*}}. \quad (3.1)$$

For every $n \in \mathbb{N}$, we set $f_n \in X = (\sum_{k=1}^{\infty} \oplus X_k)_p$ by

$$\pi_k(f_n) = \begin{cases} x_k^{(n)} & , \quad p_n \leq k \leq q_n \\ 0 & , \quad \text{otherwise} \end{cases}$$

Then the sequence $(f_n)_n$ is weakly p^* -summable. Indeed, for every $x^* \in X^*$, we have

$$\begin{aligned} |\langle x^*, f_n \rangle| &= \left| \sum_{k=p_n}^{q_n} \langle P_k x^*, x_k^{(n)} \rangle \right| \\ &\leq \left(\sum_{k=p_n}^{q_n} \|x_k^{(n)}\|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{k=p_n}^{q_n} \|P_k x^*\|^{p^*} \right)^{\frac{1}{p^*}} \\ &= \left(\sum_{k=p_n}^{q_n} \|P_k x^*\|^{p^*} \right)^{\frac{1}{p^*}}, \end{aligned}$$

which implies

$$\sum_{n=1}^{\infty} |\langle x^*, f_n \rangle|^{p^*} \leq \sum_{n=1}^{\infty} \sum_{k=p_n}^{q_n} \|P_k x^*\|^{p^*} \leq \sum_{k=1}^{\infty} \|P_k x^*\|^{p^*} < \infty.$$

By (1), we get

$$\sum_{k=p_n}^{q_n} \langle P_k x_n^*, x_k^{(n)} \rangle = |\langle x_n^*, f_n \rangle| \leq \sup_{x^* \in A} |\langle x^*, f_n \rangle| \rightarrow 0 \quad (n \rightarrow \infty),$$

which contradicts with (3.1).

(2) \Rightarrow (1). Let T be an operator from $l_p(c_0$ for $p^* = 1)$ into X . Then, by (2), we have

$$\sup_{x^* \in A} \left\| \sum_{k=1}^n T^* \circ J_k \circ P_k x^* - T^* x^* \right\| \leq \|T\| \cdot \sup_{x^* \in A} \left(\sum_{k=n+1}^{\infty} \|P_k x^*\|^{p^*} \right)^{\frac{1}{p^*}} \rightarrow 0 \quad (n \rightarrow \infty).$$

Thus, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$T^* A \subset \sum_{k=1}^{n_0} T^* \circ J_k \circ P_k A + \epsilon B_{l_{p^*}}.$$

Since $P_k(A)$ is a p^* -(V) set for each $k = 1, 2, \dots, n_0$, we get, by Theorem 2.1, that the subset $T^* \circ \pi_k^* \circ P_k A$ is relatively norm compact for each $k = 1, 2, \dots, n_0$ and so is $\sum_{k=1}^{n_0} T^* \circ J_k \circ P_k A$. Therefore, the subset $T^* A$ is relatively norm compact. Again by Theorem 2.1, we see that A is a p^* -(V) set. \square

Theorem 3.2. *Let $(X_n)_n$ be a sequence of Banach spaces, $1 < p < \infty$, $1 \leq q < p^*$ and let $X = (\sum_{n=1}^{\infty} \oplus X_n)_p$. Then a bounded subset A of X^* is a q -(V) set if and only if each $P_n(A)$ does.*

Proof. We need only prove the sufficient part. Assume that A is not a q -(V) set. Then there exist $\epsilon_0 > 0$, a sequence $(x_n)_n \in l_q^w(X)$ and a sequence $(x_n^*)_n$ in A such that

$$|\langle x_n^*, x_n \rangle| = \left| \sum_{k=1}^{\infty} \langle P_k x_n^*, \pi_k x_n \rangle \right| > \epsilon_0, \quad n = 1, 2, \dots \quad (3.2)$$

By the assumption, we get

$$\lim_{n \rightarrow \infty} \langle P_k x_n^*, \pi_k x_n \rangle = 0, \quad k = 1, 2, \dots \quad (3.3)$$

By induction on n in (3.2) and k in (3.3), we get

$$1 = n_1 < n_2 < \dots, \quad 0 = k_0 < k_1 < k_2 < \dots,$$

such that

$$\left| \sum_{k=k_j+1}^{\infty} \langle P_k x_{n_j}^*, \pi_k x_{n_j} \rangle \right| < \frac{\epsilon_0}{4}, \quad j = 1, 2, \dots \quad (3.4)$$

and

$$\left| \sum_{k=1}^{k_j} \langle P_k x_{n_{j+1}}^*, \pi_k x_{n_{j+1}} \rangle \right| < \frac{\epsilon_0}{4}, \quad j = 1, 2, \dots \quad (3.5)$$

By (3.2), (3.4) and (3.5), we get

$$\left| \sum_{k=k_{j-1}+1}^{k_j} \langle P_k x_{n_j}^*, \pi_k x_{n_j} \rangle \right| > \frac{\epsilon_0}{2}, \quad j = 2, 3, \dots$$

By (3.2) and (3.4), we get

$$\left| \sum_{k=1}^{k_1} \langle P_k x_{n_1}^*, \pi_k x_{n_1} \rangle \right| > \frac{3}{4} \epsilon_0 > \frac{\epsilon_0}{2}.$$

Thus, we have

$$\left| \sum_{k=k_{j-1}+1}^{k_j} \langle P_k x_{n_j}^*, \pi_k x_{n_j} \rangle \right| > \frac{\epsilon_0}{2}, \quad j = 1, 2, \dots$$

For each $j = 1, 2, \dots$, we set $y_j = x_{n_j}$ and $y_j^* \in X^*$ by

$$P_k y_j^* = \begin{cases} P_k x_{n_j}^* & , \quad k_{j-1} + 1 \leq k \leq k_j \\ 0 & , \quad \text{otherwise} \end{cases}$$

Clearly, $(y_j)_j \in l_q^w(X)$ and

$$\left| \langle y_j^*, y_j \rangle \right| = \left| \sum_{k=k_{j-1}+1}^{k_j} \langle P_k x_{n_j}^*, \pi_k x_{n_j} \rangle \right| > \frac{\epsilon_0}{2}, \quad j = 1, 2, \dots$$

Since the sequence $(y_j^*)_j$ has pairwise disjoint supports, we see that $(y_j^*)_j$ is equivalent to the unit vector basis $(e_j)_j$ of l_{p^*} . Let R be an isomorphic embedding from l_{p^*} into X^* with $Re_j = y_j^* (j = 1, 2, \dots)$. Let T be an any operator from l_{q^*} into X . By Pitt's Theorem, the operator T^*R is compact and hence the sequence $(T^*y_j^*)_j = (T^*Re_j)_j$ is relatively norm compact. It

follows from Theorem 2.1 that the sequence $(y_j^*)_j$ is a q -(V) set. Since $(y_j)_j$ is weakly q -summable, we have

$$| \langle y_n^*, y_n \rangle | \leq \sup_j | \langle y_j^*, y_n \rangle | \rightarrow 0 \quad (n \rightarrow \infty),$$

this contradiction concludes the proof. □

The following two lemmas are well-known (see [5], for example).

Lemma 3.3. *Let $(X_n)_n$ be a sequence of Banach spaces. The following are equivalent about a bounded subset A of $(\sum_{n=1}^{\infty} \oplus X_n)_1$:*

- (1) *A is relatively weakly compact;*
- (2) *$\pi_n(A)$ is relatively weakly compact for each $n \in \mathbb{N}$ and*

$$\lim_{n \rightarrow \infty} \sup \left\{ \sum_{k=n}^{\infty} \|\pi_k x\| : x \in A \right\} = 0.$$

Lemma 3.4. *Let $(X_n)_n$ be a sequence of Banach spaces and let $X = (\sum_{n=1}^{\infty} \oplus X_n)_p$ ($1 < p < \infty$) or $X = (\sum_{n=1}^{\infty} \oplus X_n)_0$. Then a bounded subset A of X is relatively weakly compact if and only if every $\pi_n(A)$ does.*

Theorem 3.5. *Let $(X_n)_n$ be a sequence of Banach spaces, $1 \leq q < \infty$ and $1 < p < \infty$. Then $(\sum_{n=1}^{\infty} \oplus X_n)_p$ has property q -(V) if and only if each X_n does.*

Proof. The necessary part follows from Corollary 2.4.

Conversely, let A be a q -(V) subset of $(\sum_{n=1}^{\infty} \oplus X_n^*)_{p^*}$. Then each $P_n A$ is also a q -(V) set. By hypothesis, each $P_n A$ is relatively weakly compact. It follows from Lemma 3.4 that A is relatively weakly compact. This concludes the proof. □

Theorem 3.6. *Let $(X_n)_n$ be a sequence of Banach spaces. Then $(\sum_{n=1}^{\infty} \oplus X_n)_0$ has property 1-(V) if and only if each X_n does.*

Proof. The necessary part follows from Corollary 2.4.

Conversely, assume that A is a 1-(V) subset of $(\sum_{n=1}^{\infty} \oplus X_n^*)_1$. By Theorem 3.1, each $P_n(A)$ is a 1-(V) set and

$$\lim_{n \rightarrow \infty} \sup \left\{ \sum_{k=n}^{\infty} \|P_k x^*\| : x^* \in A \right\} = 0.$$

By the assumption, each $P_n A$ is relatively weakly compact. It follows from Lemma 3.3 that A is relatively weakly compact. We are done. \square

Combining Theorem 2.9 with the same argument as Theorem 3.1, we obtain the similar result for the p -(V^*) sets.

Theorem 3.7. *Let $(X_n)_n$ be a sequence of Banach spaces. Let A be a bounded subset of $X = (\sum_{n=1}^{\infty} \oplus X_n)_p$ ($1 \leq p < \infty$). The following assertions are equivalent:*

- (1) A is a p -(V^*) set;
- (2) $\pi_n(A)$ is a p -(V^*) set for each $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \sup \left\{ \sum_{k=n}^{\infty} \|\pi_k x\|^p : x \in A \right\} = 0.$$

Theorem 3.8. *Let $(X_n)_n$ be a sequence of Banach spaces and $1 \leq q < p < \infty$. Let A be a bounded subset of $X = (\sum_{n=1}^{\infty} \oplus X_n)_p$ or $X = (\sum_{n=1}^{\infty} \oplus X_n)_0$. Then A is a q -(V^*) set if and only if each $\pi_n(A)$ does.*

The proof is similar to Theorem 3.2, only interchanging the role of X and X^* and replacing Theorem 2.1 by Theorem 2.9.

Theorem 3.9. *Let $(X_n)_n$ be a sequence of Banach spaces, $1 \leq q < \infty$, $1 < p < \infty$ and let $X = (\sum_{n=1}^{\infty} \oplus X_n)_p$ or $X = (\sum_{n=1}^{\infty} \oplus X_n)_0$. Then X has property q -(V^*) if and only if each X_n does.*

Proof. The necessary part follows from Corollary 2.13.

Conversely, let A be a q -(V^*) subset of X . Clearly, each $\pi_n(A)$ is a q -(V^*) subset of X_n . By the assumption, each $\pi_n(A)$ is relatively weakly compact. It follows from Lemma 3.4 that A is relatively weakly compact. Therefore, X has property q -(V^*). \square

Theorem 3.10. *Let $(X_n)_n$ be a sequence of Banach spaces and $1 \leq p < \infty$. Then $(\sum_{n=1}^{\infty} \oplus X_n)_p$ has property p -(V^*) if and only if so does each X_n .*

Proof. The necessary part follows from Corollary 2.13.

Conversely, let A be a p -(V^*) set. It follows from Theorem 3.7 that each $\pi_n(A)$ is a p -(V^*) set and

$$\lim_{n \rightarrow \infty} \sup \left\{ \sum_{k=n}^{\infty} \|\pi_k x\|^p : x \in A \right\} = 0.$$

Since each X_n has property p -(V^*), each $\pi_n(A)$ is relatively weakly compact. For $1 < p < \infty$, it follows from Lemma 3.4 that A is relatively weakly compact. For $p = 1$, Lemma 3.3 yields that A is relatively weakly compact. Thus, in both cases, A is relatively weakly compact. This concludes the proof. \square

4. Quantifying Pełczyński's property (V) of order p and Pełczyński's property (V^*) of order p

We will need several measures of weak non-compactness. Let A be a bounded subset of a Banach space X . Other commonly used quantities measuring weak non-compactness are:

$$wk_X(A) = \widehat{d}(\overline{A}^{w*}, X), \text{ where } \overline{A}^{w*} \text{ denotes the } weak^* \text{ closure of } A \text{ in } X^{**}.$$

$$wck_X(A) = \sup \{ d(clust_{X^{**}}((x_n)_n), X) : (x_n)_n \text{ is a sequence in } A \}, \text{ where } clust_{X^{**}}((x_n)_n) \text{ is the set of all } weak^* \text{ cluster points in } X^{**} \text{ of } (x_n)_n.$$

$$\gamma_X(A) = \sup \{ | \lim_n \lim_m \langle x_m^*, x_n \rangle - \lim_m \lim_n \langle x_m^*, x_n \rangle | : (x_n)_n \text{ is a sequence in } A, (x_m^*)_m \text{ is a sequence in } B_{X^*} \text{ and all the involved limits exist} \}.$$

It follows from [3, Theorem 2.3] that for any bounded subset A of a Banach space X we have

$$wck_X(A) \leq wk_X(A) \leq \gamma_X(A) \leq 2wck_X(A), \quad (4.1)$$

$$wk_X(A) \leq \omega(A).$$

For an operator $T : X \rightarrow Y$, $\omega(T)$, $wk_Y(T)$, $wck_Y(T)$, $\gamma_Y(T)$ will denote $\omega(TB_X)$, $wk_Y(TB_X)$, $wck_Y(TB_X)$ and $\gamma_Y(TB_X)$, respectively. C. Angosto and B. Cascales([3])proved the following inequality:

$$\gamma_Y(T) \leq \gamma_{X^*}(T^*) \leq 2\gamma_Y(T), \text{ for any operator } T.$$

So, putting these inequalities together, we get, for any operator T ,

$$\frac{1}{2}wk_Y(T) \leq wk_{X^*}(T^*) \leq 4wk_Y(T). \quad (4.2)$$

For an operator $T : X \rightarrow Y$ and $1 \leq p \leq \infty$. We set

$$uc_p(T) = \sup\{\limsup_n \|Tx_n\| : (x_n)_n \in l_p^w(X), \|(x_n)_n\|_p^w \leq 1\},$$

We begin this section with a simple lemma in [7], which will be used frequently.

Lemma 4.1. [7] *Let X be a closed subspace of a Banach space Y and let A be a bounded subset of X . Then*

$$wk_Y(A) \leq wk_X(A) \leq 2wk_Y(A). \quad (4.3)$$

It is worth mentioning that the constant 2 in the right inequality of (4.3) is optimal. Indeed, let $X = c_0, Y = l_\infty$ and A be the summing basis of c_0 . It is easy to check that $wk_X(A) = 1$ and $wk_Y(A) = \frac{1}{2}$.

Definition 4.1. Let $1 \leq p \leq \infty$. We say that a Banach space X has *quantitative Pełczyński's property (V^*) of order p* (property $p\text{--}(V^*)_q$ in short) with a constant $C > 0$ if for every Banach space Y and every operator $T : Y \rightarrow X$, one has

$$wk_X(T) \leq C \cdot uc_p(T^*).$$

We say that a Banach space X has property $p\text{--}(V^*)_q$ if it has property $p\text{--}(V^*)_q$ with some constant C .

Let $1 \leq p \leq \infty$ and X be a Banach space. For a bounded subset A of X , we set

$$\iota_p(A) = \sup\{\limsup_n \sup_{x \in A} |\langle x_n^*, x \rangle| : (x_n^*)_n \in l_p^w(X^*), \|(x_n^*)_n\|_p^w \leq 1\}.$$

Theorem 4.2. *Let X be a Banach space and $1 \leq p < \infty$. The following statements are equivalent:*

- (1) X has property $p\text{--}(V^*)_q$;

(2) there exists a constant $C > 0$ such that for each bounded subset A of X , one has

$$wk_X(A) \leq C \cdot \iota_p(A).$$

Proof. (1) \Rightarrow (2). Suppose that X has property $p-(V^*)_q$ with a constant $C > 0$. Let A be a bounded subset of X . We first claim that

$$wck_X(A) \leq C \cdot \iota_p(A).$$

Indeed, we may assume that $wck_X(A) > 0$ and fix an arbitrary $\epsilon \in (0, wck_X(A))$. Then there exists a sequence $(x_n)_n$ in A such that $\epsilon < wck_X((x_n)_n)$. Define an operator

$$T : l_1 \rightarrow X, \quad (\alpha_n)_n \mapsto \sum_{n=1}^{\infty} \alpha_n x_n, \quad (\alpha_n)_n \in l_1.$$

By (1) and (4.1), we get

$$wck_X((x_n)_n) \leq wck_X(T) \leq wk_X(T) \leq C \cdot uc_p(T^*).$$

By the definitions of T and $\iota_p(A)$, we get

$$uc_p(T^*) \leq \iota_p(A).$$

This yields $\epsilon < C \cdot \iota_p(A)$. By the arbitrariness of $\epsilon \in (0, wck_X(A))$, we prove the claim. Again by (4.1), we obtain

$$wk_X(A) \leq 2C \cdot \iota_p(A).$$

(2) \Rightarrow (1) is trivial by taking $A = TB_Y$ with the same constant C . □

Theorem 4.3. *Let $1 \leq p \leq \infty$. If a Banach space X has property $p-(V^*)_q$ with a constant C , then every closed subspace of X has property $p-(V^*)_q$ with $2C$.*

Proof. Let M be a closed subspace of X . Suppose that X has property $p-(V^*)_q$ with a constant $C > 0$. We'll show that M has property $p-(V^*)_q$ with $2C$. Fix a Banach space Y and an operator $S : Y \rightarrow M$. Then we have

$$wk_X(iS) \leq C \cdot uc_p(S^*i^*),$$

where $i : M \rightarrow X$ is the inclusion map. By (4.3), we get

$$wk_M(S) \leq 2wk_X(iS) \leq 2C \cdot uc_p(S^*i^*) \leq 2C \cdot uc_p(S^*),$$

we are done. □

Theorem 4.4. *Let $1 \leq p \leq \infty$. Then a Banach space X has property p -(V^*) $_q$ if and only if there exists a constant $C > 0$ such that every separable closed subspace of X has property p -(V^*) $_q$ with C .*

Proof. The necessary part follows from Theorem 4.3. Conversely, let $C > 0$ be such that every separable closed subspace of X has property p -(V^*) $_q$ with C . We claim that

$$\gamma_X(T) \leq 2C \cdot uc_p(T^*), \text{ for every Banach space } Y \text{ and every operator } T : Y \rightarrow X.$$

Fix a space Y and an operator $T : Y \rightarrow X$. Let $A = TB_Y$. We may assume that $\gamma_X(A) > 0$. Fix any $\epsilon \in (0, \gamma_X(A))$. Then there exists a sequence $(x_n)_n$ in A such that $\epsilon < \gamma_X((x_n)_n)$. By [15, Proposition 3.4], there exist a separable closed subspace Z of X that contains $(x_n)_n$ and an isometric embedding $J : Z^* \rightarrow X^*$ such that $Jz^*|_Z = z^*$ for every $z^* \in Z^*$. Define an operator

$$P : X^* \rightarrow J(Z^*), \quad x^* \mapsto J(x^*|_Z), \quad x^* \in X^*.$$

Then P is a linear projection from X^* onto $J(Z^*)$ with $\|P\| = 1$. We define an operator

$$S : l_1 \rightarrow Z, \quad (\alpha_n)_n \mapsto \sum_{n=1}^{\infty} \alpha_n x_n, \quad (\alpha_n)_n \in l_1.$$

By hypothesis, we get

$$wk_Z(S) \leq C \cdot uc_p(S^*).$$

By (4.1), we have

$$wk_Z(S) \geq wk_Z((x_n)_n) \geq \frac{1}{2} \gamma_Z((x_n)_n).$$

By the definition of S and the properties of J , we obtain

$$\begin{aligned}\gamma_Z((x_n)_n) &\leq 2C \cdot \sup\{\limsup_n \sup_k | \langle z_n^*, x_k \rangle | : (z_n^*)_n \in l_p^w(Z^*), \|(z_n^*)_n\|_p^w \leq 1\} \\ &\leq 2C \cdot \sup\{\limsup_n \sup_k | \langle x_n^*, x_k \rangle | : (x_n^*)_n \in l_p^w(X^*), \|(x_n^*)_n\|_p^w \leq 1\} \\ &\leq 2C \cdot uc_p(T^*)\end{aligned}$$

By the definition of P , we get

$$\gamma_X((x_n)_n) = \gamma_Z((x_n)_n).$$

Thus, one has $\epsilon < 2C \cdot uc_p(T^*)$, which proves the claim by the arbitrariness of ϵ .

By (4.1) again, we have

$$wk_X(T) \leq 2C \cdot uc_p(T^*).$$

This implies that X has property $p\text{-}(V^*)_q$ with $2C$.

□

Theorem 4.5. *Let $X = L_1(\mu, \mathbb{R})$, where (Ω, Σ, μ) is a finite measure space. Then*

$$wk_X(A) = \iota_1(A)$$

for each bounded subset A of X .

Proof. We may assume that A is a subset of B_X .

Step 1. $wk_X(A) \leq \iota_1(A)$.

Let us assume that $wk_X(A) > 0$ and fix an arbitrary $\epsilon \in (0, wk_X(A))$. It follows from [17, Proposition 7.1] that there exists a sequence $(x_k)_k$ in A such that

$$\int_{E_k} |x_k| d\mu > \epsilon, \quad k = 1, 2, \dots \quad (4.4)$$

where $E_k = \{t \in \Omega : |x_k(t)| \geq k\}$.

For each $k \in \mathbb{N}$, Chebyshev's inequality gives $\mu(E_k) \leq \frac{1}{k}$ and hence the sequence $(x_k \chi_{E_k})_k$ converges to 0 in measure. By [4, Lemma 5.2.1], there exist a subsequence $(x_{k_n} \chi_{E_{k_n}})_n$ of $(x_k \chi_{E_k})_k$ and a sequence of disjoint measurable sets $(A_n)_n$ such that

$$\|x_{k_n} \chi_{E_{k_n}} - x_{k_n} \chi_{E_{k_n}} \chi_{A_n}\|_1 \rightarrow 0 \quad (n \rightarrow \infty). \quad (4.5)$$

Set $B_n = E_{k_n} \cap A_n$ and $f_n = \text{sign}(x_{k_n})\chi_{B_n}$ for each $n \in \mathbb{N}$. Then $(f_n)_n$ is weakly 1-summable in X^* and $\|(f_n)_n\|_1^w \leq 1$. Combining (4.4) with (4.5), we get

$$\limsup_n | \langle f_n, x_{k_n} \rangle | \geq \epsilon,$$

which implies that $\iota_1(A) \geq \epsilon$. Since $\epsilon \in (0, wk_X(A))$ is arbitrary, we conclude Step 1.

Step 2. $\iota_1(A) \leq wk_X(A)$.

Similarly, we can assume that $\iota_1(A) > 0$ and fix an arbitrary $\epsilon \in (0, \iota_1(A))$. Then there exist a sequence $(x_n)_n$ in A and $(f_n)_n \in l_1^w(X^*)$ with $\|(f_n)_n\|_1^w \leq 1$ such that

$$\int_{\Omega} f_n \cdot x_n d\mu > \epsilon, \quad n = 1, 2, \dots \quad (4.6)$$

It is easy to verify that

$$\|(f_n)_n\|_1^w = \sup_n \left\| \sum_{k=1}^n |f_k| \right\|.$$

Thus, we get

$$\sum_{k=1}^{\infty} |f_k| \leq 1, \mu - a.e.$$

For the sake of convenience, we may assume that $\sum_{k=1}^{\infty} |f_k| \leq 1$ everywhere.

Let $\delta > 0$ be arbitrary. By (4.6), we obtain $N_1 \in \mathbb{N}$ such that

$$\int_{E_1} f_1 \cdot x_1 d\mu > \epsilon,$$

where $E_1 = \{t \in \Omega : \sum_{k=N_1}^{\infty} |f_k(t)| < \frac{\delta}{2}\}$.

Set $\tilde{f}_1 = f_1 \cdot \chi_{E_1}$ and $\tilde{f}_n = f_n \cdot \chi_{E_1^c}$ ($n \geq N_1$). Then \tilde{f}_1 is disjoint from \tilde{f}_n for each $n \geq N_1$. Moreover, for each $n \geq N_1$, we have

$$\int_{\Omega} \tilde{f}_n \cdot x_n d\mu = \int_{\Omega} f_n \cdot x_n d\mu - \int_{E_1} f_n \cdot x_n d\mu > \epsilon - \frac{\delta}{2}. \quad (4.7)$$

Obviously,

$$\sum_{k=N_1}^{\infty} |\tilde{f}_k| \leq \sum_{k=N_1}^{\infty} |f_k| \leq 1 \text{ everywhere.}$$

By (4.7), we obtain $N_2 > N_1$ such that

$$\int_{E_2} \tilde{f}_{N_1} \cdot x_{N_1} d\mu > \epsilon - \frac{\delta}{2}. \quad (4.8)$$

where $E_2 = \{t \in \Omega : \sum_{k=N_2}^{\infty} |\tilde{f}_k(t)| < \frac{\delta}{2^2}\}$.

Set $\tilde{\tilde{f}}_2 = \tilde{f}_{N_1} \cdot \chi_{E_2}$ and $\tilde{f}_n = \tilde{f}_n \cdot \chi_{E_2^c} (n \geq N_2)$. Then, by (4.8), we get

$$\int_{F_1} |x_{N_1}| d\mu \geq \int_{\Omega} \tilde{\tilde{f}}_2 \cdot x_{N_1} d\mu = \int_{E_2} \tilde{f}_{N_1} \cdot x_{N_1} d\mu > \epsilon - \frac{\delta}{2}. \quad (4.9)$$

where F_1 is the support of $\tilde{\tilde{f}}_2$.

By (4.7), for each $n \geq N_2$, we have

$$\begin{aligned} \int_{\Omega} \tilde{\tilde{f}}_n \cdot x_n d\mu &= \int_{E_2^c} \tilde{f}_n \cdot x_n d\mu \\ &= \int_{\Omega} \tilde{f}_n \cdot x_n d\mu - \int_{E_2} \tilde{f}_n \cdot x_n d\mu \\ &> \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2}. \end{aligned}$$

In a similar way, we obtain $N_3 > N_2$ such that

$$\int_{E_3} \tilde{\tilde{f}}_{N_2} \cdot x_{N_2} d\mu > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2}. \quad (4.10)$$

where $E_3 = \{t \in \Omega : \sum_{k=N_3}^{\infty} |\tilde{\tilde{f}}_k(t)| < \frac{\delta}{2^3}\}$.

Set $\tilde{\tilde{\tilde{f}}}_3 = \tilde{\tilde{f}}_{N_2} \cdot \chi_{E_3}$ and $\tilde{\tilde{f}}_n = \tilde{\tilde{f}}_n \cdot \chi_{E_3^c} (n \geq N_3)$. Then, by (4.10), we get

$$\int_{F_2} |x_{N_2}| d\mu \geq \int_{\Omega} \tilde{\tilde{\tilde{f}}}_3 \cdot x_{N_2} d\mu > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2}. \quad (4.11)$$

where F_2 is the support of $\tilde{\tilde{\tilde{f}}}_3$.

Since $\tilde{\tilde{f}}_2$ is disjoint from $\tilde{\tilde{f}}_n$ for each $n \geq N_2$, the set F_2 is also disjoint from F_1 .

A similar computation shows that for each $n \geq N_3$,

$$\int_{\Omega} \tilde{\tilde{\tilde{f}}}_n \cdot x_n d\mu > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2} - \frac{\delta}{2^3}.$$

Thus, there exists $N_4 > N_3$ such that

$$\int_{E_4} \tilde{\tilde{f}}_{N_3} \cdot x_{N_3} d\mu > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2} - \frac{\delta}{2^3}. \quad (4.12)$$

where $E_4 = \{t \in \Omega : \sum_{k=N_4}^{\infty} |\tilde{\tilde{f}}_k(t)| < \frac{\delta}{2^4}\}$.

Set $\tilde{\tilde{f}}_4 = \tilde{\tilde{f}}_{N_3} \cdot \chi_{E_4}$ and $\tilde{\tilde{f}}_n = \tilde{\tilde{f}}_n \cdot \chi_{E_4^c}$ for $n \geq N_4$. Inequality (4.12) yields

$$\int_{F_3} |x_{N_3}| d\mu \geq \int_{\Omega} \tilde{\tilde{f}}_4 \cdot x_{N_3} d\mu > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2} - \frac{\delta}{2^3},$$

where F_3 is the support of $\tilde{\tilde{f}}_4$.

Since $\tilde{\tilde{f}}_3$ is disjoint from $\tilde{\tilde{f}}_{N_3}$, the set F_3 is disjoint from F_2 . Since $\tilde{\tilde{f}}_2$ is disjoint from $\tilde{\tilde{f}}_{N_3}$, the set F_3 is disjoint from F_1 .

By induction, we get a subsequence $(x_{N_k})_k$ of $(x_n)_n$ and a sequence of disjoint measurable sets $(F_k)_k$ such that

$$\int_{F_k} |x_{N_k}| d\mu > \epsilon - \sum_{i=1}^k \frac{\delta}{2^i} > \epsilon - \delta, \quad k = 1, 2, \dots$$

Claim: $wk_X(A) \geq \epsilon - \delta$.

Indeed, fix any $c > 0$. Then, for each $k \in \mathbb{N}$,

$$\begin{aligned} \epsilon - \delta &< \int_{F_k} |x_{N_k}| d\mu \\ &= \int_{F_k \cap \{|x_{N_k}| \geq c\}} (|x_{N_k}| - c) d\mu + c\mu(F_k \cap \{|x_{N_k}| \geq c\}) + \int_{F_k \cap \{|x_{N_k}| < c\}} |x_{N_k}| d\mu \\ &\leq \int_{F_k \cap \{|x_{N_k}| \geq c\}} (|x_{N_k}| - c) d\mu + c\mu(F_k \cap \{|x_{N_k}| \geq c\}) + c\mu(F_k \cap \{|x_{N_k}| < c\}) \\ &\leq \int_{\{|x_{N_k}| \geq c\}} (|x_{N_k}| - c) d\mu + c\mu(F_k) \\ &\leq \sup_{x \in A} \int_{\{|x| \geq c\}} (|x| - c) d\mu + c\mu(F_k) \end{aligned}$$

Since $(F_k)_k$ is disjoint and μ is finite, $\mu(F_k) \rightarrow 0 (k \rightarrow \infty)$.

Letting $k \rightarrow \infty$, we get

$$\epsilon - \delta \leq \sup_{x \in A} \int_{\{|x| \geq c\}} (|x| - c) d\mu.$$

Again by [17, Proposition 7.1], we prove the claim.

Since $\delta > 0$ is arbitrary, we get

$$wk_X(A) \geq \epsilon.$$

By the arbitrariness of $\epsilon \in (0, \iota_1(A))$, we obtain

$$\iota_1(A) \leq wk_X(A).$$

This completes the proof. □

Theorem 4.6. *Let $X = l_1(\mathbb{N}, \mathbb{R})$. Then*

$$wk_X(A) = \iota_1(A)$$

for each bounded subset A of X .

Proof. We may assume that A is a subset of B_X .

Step 1. $wk_X(A) \leq \iota_1(A)$.

We may assume that $wk_X(A) > 0$ and fix an arbitrary $c \in (0, wk_X(A))$. By [17, Lemma 7.2], we obtain two sequences $(p_n)_n, (q_n)_n$ of natural numbers with $p_n < q_n < p_{n+1} (n \in \mathbb{N})$ and a sequence $(x_n)_n$ in A such that

$$\sum_{k=p_n}^{q_n} |x_n(k)| > c, \quad n = 1, 2, \dots$$

By Hahn-Banach Theorem, for each $n \in \mathbb{N}$, we can find $(\alpha_n(k))_{k=p_n}^{q_n}$ with $\sup_{p_n \leq k \leq q_n} |\alpha_n(k)| = 1$ such that

$$\sum_{k=p_n}^{q_n} x_n(k) \alpha_n(k) > c. \tag{4.13}$$

For each $n \in \mathbb{N}$, we set $f_n \in l_\infty$ by

$$f_n(k) = \begin{cases} \alpha_n(k) & , \quad p_n \leq k \leq q_n \\ 0 & , \quad otherwise \end{cases}$$

Then $(f_n)_n$ is weakly 1-summable and $\|(f_n)_n\|_1^w \leq 1$. Indeed, for each $x \in X$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} | \langle f_n, x \rangle | &= \sum_{n=1}^{\infty} \left| \sum_{k=p_n}^{q_n} x(k) \alpha_n(k) \right| \\ &\leq \sum_{n=1}^{\infty} \sum_{k=p_n}^{q_n} |x(k)| |\alpha_n(k)| \\ &\leq \sum_{n=1}^{\infty} \sum_{k=p_n}^{q_n} |x(k)| \leq \|x\|. \end{aligned}$$

It follows from (4.13) that

$$\sup_{x \in A} | \langle f_n, x \rangle | \geq | \langle f_n, x_n \rangle | = \sum_{k=p_n}^{q_n} x_n(k) \alpha_n(k) > c, \quad n = 1, 2, \dots$$

This implies that $\iota_1(A) \geq c$.

Since $c \in (0, wk_X(A))$ is arbitrary, we get

$$wk_X(A) \leq \iota_1(A).$$

Step 2. $\iota_1(A) \leq wk_X(A)$.

Assume that $\iota_1(A) > 0$ and fix an arbitrary $\epsilon \in (0, \iota_1(A))$. Then there exist a sequence $(x_n)_n$ in A and $(f_n)_n \in l_1^w(X^*)$ with $\|(f_n)_n\|_1^w \leq 1$ such that

$$\sum_{k=1}^{\infty} f_n(k) x_n(k) > \epsilon, \quad n = 1, 2, \dots \quad (4.14)$$

It follows from $\|(f_n)_n\|_1^w \leq 1$ that

$$\sum_{n=1}^{\infty} |f_n(k)| \leq 1, \quad k = 1, 2, \dots \quad (4.15)$$

Let $\delta > 0$ be arbitrary. By (4.14) and (4.15), there exists $N_1 \in \mathbb{N}$ such that

$$\sum_{k \in E_1} f_1(k)x_1(k) > \epsilon,$$

where $E_1 = \{k : \sum_{n=N_1}^{\infty} |f_n(k)| < \frac{\delta}{2}\}$.

Set $\tilde{f}_1 = f_1 \cdot \chi_{E_1}$ and $\tilde{f}_n = f_n \cdot \chi_{E_1^c}$ ($n \geq N_1$). By (4.14) and the definition of E_1 , we have

$$\langle \tilde{f}_n, x_n \rangle = \sum_{k=1}^{\infty} f_n(k)x_n(k) - \sum_{k \in E_1} f_n(k)x_n(k) > \epsilon - \frac{\delta}{2}, \quad (4.16)$$

for each $n \geq N_1$.

Inequality (4.15) yields that

$$\sum_{n=N_1}^{\infty} |\tilde{f}_n(k)| \leq \sum_{n=N_1}^{\infty} |f_n(k)| \leq 1, \quad k = 1, 2, \dots \quad (4.17)$$

Combining (4.16) with (4.17), we obtain $N_2 > N_1$ such that

$$\sum_{k \in E_2} \tilde{f}_{N_1}(k)x_{N_1}(k) > \epsilon - \frac{\delta}{2}, \quad (4.18)$$

where $E_2 = \{k : \sum_{n=N_2}^{\infty} |\tilde{f}_n(k)| < \frac{\delta}{2^2}\}$.

Set $\tilde{\tilde{f}}_2 = \tilde{f}_{N_1} \cdot \chi_{E_2}$ and $\tilde{\tilde{f}}_n = \tilde{f}_n \cdot \chi_{E_2^c}$ ($n \geq N_2$). Then, by (4.18), we get

$$\sum_{k \in F_1} |x_{N_1}(k)| \geq \sum_{k=1}^{\infty} \tilde{\tilde{f}}_2(k)x_{N_1}(k) > \epsilon - \frac{\delta}{2},$$

where F_1 is the support of $\tilde{\tilde{f}}_2$.

Moreover, by (4.16) and the definition of E_2 , we get

$$\langle \tilde{\tilde{f}}_n, x_n \rangle = \sum_{k=1}^{\infty} \tilde{\tilde{f}}_n(k)x_n(k) - \sum_{k \in E_2} \tilde{\tilde{f}}_n(k)x_n(k) > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2}, \quad (4.19)$$

for each $n \geq N_2$.

By (4.19), there exists $N_3 > N_2$ such that

$$\sum_{k \in E_3} \widetilde{\widetilde{f}}_{N_2}(k) x_{N_2}(k) > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2}, \quad (4.20)$$

where $E_3 = \{k : \sum_{n=N_3}^{\infty} |\widetilde{\widetilde{f}}_n(k)| < \frac{\delta}{2^3}\}$.

Set $\widetilde{\widetilde{f}}_3 = \widetilde{\widetilde{f}}_{N_2} \cdot \chi_{E_3}$ and $\widetilde{\widetilde{f}}_n = \widetilde{\widetilde{f}}_n \cdot \chi_{E_3^c}(n \geq N_3)$. Then, by (4.20), we get

$$\sum_{k \in F_2} |x_{N_2}(k)| \geq \sum_{k=1}^{\infty} \widetilde{\widetilde{f}}_3(k) x_{N_2}(k) > \epsilon - \frac{\delta}{2} - \frac{\delta}{2^2},$$

where F_2 is the support of $\widetilde{\widetilde{f}}_3$.

Since $\widetilde{\widetilde{f}}_2$ is disjoint from $\widetilde{\widetilde{f}}_{N_2}$, the set F_2 is disjoint from F_1 .

As in the proof of Theorem 4.5, we get a subsequence $(x_{N_k})_k$ of $(x_n)_n$ and a sequence $(F_k)_k$ of pairwise disjoint subsets of \mathbb{N} such that

$$\sum_{i \in F_k} |x_{N_k}(i)| > \epsilon - \sum_{i=1}^k \frac{\delta}{2^i} > \epsilon - \delta, \quad k = 1, 2, \dots$$

Finally, we claim: $wk_X(A) \geq \epsilon - \delta$.

Let us fix $n \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, we have

$$\begin{aligned} \epsilon - \delta &< \sum_{i=1}^{\infty} |x_{N_k}(i)| \chi_{F_k}(i) \\ &= \sum_{i=1}^n |x_{N_k}(i)| \chi_{F_k}(i) + \sum_{i=n+1}^{\infty} |x_{N_k}(i)| \chi_{F_k}(i) \\ &\leq \sum_{i=1}^n \chi_{F_k}(i) + \sup_{x \in A} \sum_{i=n+1}^{\infty} |x(i)|. \end{aligned}$$

Since $(F_k)_k$ is disjoint pairwise, $\sum_{i=1}^n \chi_{F_k}(i) = 0$ for k large enough.

This implies

$$\epsilon - \delta \leq \inf_n \sup_{x \in A} \sum_{i=n+1}^{\infty} |x(i)|.$$

It follows from [17, Proposition 7.3] that

$$\epsilon - \delta \leq wk_X(A).$$

Since $\delta > 0$ is arbitrary, we get

$$\epsilon \leq wk_X(A).$$

The arbitrariness of $\epsilon \in (0, \iota_1(A))$ concludes the proof. \square

Definition 4.2. Let $1 \leq p \leq \infty$. We say that a Banach space X has *quantitative Pełczyński's property (V) of order p* (property $p\text{-}(V)_q$ in short) with a constant $C > 0$ if for every Banach space Y and every operator $T : X \rightarrow Y$, one has

$$wk_{X^*}(T^*) \leq C \cdot uc_p(T).$$

If a Banach space X has property $p\text{-}(V)_q$ with some constant $C > 0$, we say that X has property $p\text{-}(V)_q$.

Clearly, if a Banach space X has property $p\text{-}(V)_q$, then it has property $p\text{-}(V)$.

Let $1 \leq p \leq \infty$ and X be a Banach space. For a bounded subset A of X^* , we set

$$\eta_p(A) = \sup \left\{ \limsup_n \sup_{x^* \in A} | \langle x^*, x_n \rangle | : (x_n)_n \in l_p^w(X), \|(x_n)_n\|_p^w \leq 1 \right\}.$$

The proof of the following theorem is similar to that of Theorem 4.2.

Theorem 4.7. *Let X be a Banach space and $1 \leq p < \infty$. The following statements are equivalent:*

- (1) X has property $p\text{-}(V)_q$;
- (2) there exists a constant $C > 0$ such that for each bounded subset A of X^* , one has

$$wk_{X^*}(A) \leq C \cdot \eta_p(A).$$

Theorem 4.8. *Let $1 \leq p \leq \infty$. If a Banach space X has property $p\text{-}(V)_q$ with a constant C , then every quotient of X has property $p\text{-}(V)_q$ with $2C$.*

Proof. Let M be a closed subspace of X . Suppose that X has property p -(V) $_q$ with a constant $C > 0$. We'll show that the quotient X/M has property p -(V) $_q$ with $2C$. Fix a Banach space Y and an operator $S : X/M \rightarrow Y$. Then we have

$$wk_{X^*}(Q^*S^*) \leq C \cdot uc_p(SQ),$$

where $Q : X \rightarrow X/M$ is the canonical quotient map. By (4.3), we get

$$wk_{(X/M)^*}(S^*) \leq 2wk_{X^*}(Q^*S^*) \leq 2C \cdot uc_p(SQ) \leq 2C \cdot uc_p(S),$$

which completes the proof. □

Theorem 4.9. *Let X be a Banach space and $1 \leq p \leq \infty$. Then*

- (1) *If X has property p -(V) $_q$ with a constant C , then X^* has property p -(V^*) $_q$ with the same constant C ;*
- (2) *If X^* has property p -(V) $_q$ with a constant C , then X has property p -(V^*) $_q$ with $2C$.*

Proof. (1) Let Z be a Banach space and S be an operator from Z into X^* . Applying the assumption to S^*J_X , we get

$$wk_{X^*}(J_X^*S^{**}) \leq C \cdot uc_p(S^*J_X).$$

This yields

$$wk_{X^*}(S) = wk_{X^*}(J_X^*S^{**}J_Z) \leq wk_{X^*}(J_X^*S^{**}) \leq C \cdot uc_p(S^*J_X) \leq C \cdot uc_p(S^*),$$

which completes the proof of (1).

The assertion (2) follows immediately from (1) and Theorem 4.3. □

Let us mention that the converse of Theorem 4.9 is false. Indeed, $X = (\sum_{n=1}^{\infty} \oplus l_{\infty}^n)_1$ enjoys property 1-(V^*) $_q$ with 1 that follows from the following Theorem 4.10. But X^* fails property p -(V) for each $1 \leq p < \infty$ because X^* contains a 1-complemented subspace isometric to l_1 and l_1 fails property p -(V) for each $1 \leq p < \infty$ by Theorem 2.1.

Theorem 4.10. *Let $X = (\sum_{\gamma \in \Gamma} \oplus X_{\gamma})_1$ (Γ an infinite set), where each X_{γ} is reflexive. Then*

$$wk_X(A) \leq \iota_1(A)$$

for each bounded subset A of X .

Proof. Let A be a bounded subset of X . We may assume that $wk_X(A) > 0$ and fix an arbitrary $c \in (0, wk_X(A))$. By [17, Lemma 7.2], we obtain, by induction, a sequence $(F_n)_n$ of pairwise disjoint finite subsets of Γ and a sequence $(x_n)_n$ in A such that

$$\sum_{\gamma \in F_n} \|x_n(\gamma)\| > c, \quad n = 1, 2, \dots$$

By Hahn-Banach Theorem, for each $n \in \mathbb{N}$, there exists $(f_n(\gamma))_{\gamma \in F_n} \in (\sum_{\gamma \in F_n} \oplus X_\gamma^*)_\infty$ with $\max_{\gamma \in F_n} \|f_n(\gamma)\| = 1$ such that

$$\sum_{\gamma \in F_n} \langle f_n(\gamma), x_n(\gamma) \rangle > c.$$

For each $n \in \mathbb{N}$, define $\varphi_n \in (\sum_{\gamma \in \Gamma} \oplus X_\gamma^*)_\infty$ by

$$\varphi_n(\gamma) = \begin{cases} f_n(\gamma) & , \quad \gamma \in F_n \\ 0 & , \quad otherwise \end{cases}$$

Since $(F_n)_n$ is pairwise disjoint, it is easy to verify that $(\varphi_n)_n$ is weakly 1-summable and $\|(\varphi_n)_n\|_1^w \leq 1$. Moreover, for each $n \in \mathbb{N}$,

$$\sup_{x \in A} |\langle \varphi_n, x \rangle| \geq |\langle \varphi_n, x_n \rangle| = \sum_{\gamma \in F_n} \langle f_n(\gamma), x_n(\gamma) \rangle > c.$$

This yields that $\iota_1(A) \geq c$. Since $c \in (0, wk_X(A))$ is arbitrary, we get the conclusion. □

Theorem 4.11. *Let $X = c_0(\mathbb{N}, \mathbb{R})$. Then*

$$wk_X(A) = \eta_1(A)$$

for each bounded subset A of X^ .*

The proof is essentially analogous to Theorem 4.6, only interchanging the role of X and X^* .

H. Krulišová proved in [19] that $C_0(\Omega)$ has property $1-(V)_q$ with constant π (constant 2 in the real case) for every locally compact space Ω .

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